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NOTE ON TRANSITIVE TRANSFORMATIONS

BY JOHN C. OXTOBY*

HARVARD UNIVERSITY

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A topological transformation T of a complete, separable, metric space Ω, without isolated points, is said to be transitive if any one of the following equivalent conditions is fulfilled:

(*a*) The complete sequence of images of some point under iteration of T and T^{-1} is dense in Ω.

(*b*) The images of some point under iteration of T, and also its images under T^{-1}, are dense in Ω.

(*c*) Any neighborhood overlaps any other one under some iterate of T.

(*d*) Ω cannot be divided into two disjoint invariant subsets both containing interior points.

(*e*) If E is any invariant set having the property of Baire,[1] then either E or its complement is of first category.

This type of transitivity has been considered, for example, by Birkhoff.[2] It is sometimes called regional or topological transitivity to distinguish it from metric transitivity, for which the characteristic property is:

(*e'*) If E is any invariant measurable set, then either E or its complement is of measure zero.

In view of the analogy between measurability and sets of measure zero on the one hand, and the property of Baire and sets of first category on the other,[3] the parallelism of conditions (*e*) and (*e'*) indicates that the above type of transitivity is to be regarded as the natural topological analog of metric transitivity.

Such transformations are of interest in connection with dynamical systems. If the flow induced in the phase space by the equations of the system admits a surface of section, then a necessary and sufficient condition that there exist everywhere dense streamlines is that the transformation of the surface of section be transitive.[2] Transformations which arise in this way from Lagrangean systems admit an invariant integral, so that measure-preserving transformations are of special interest.[4] The existence of dense streamlines in the case of geodesic motion on surfaces of genus $p > 1$ has been shown by Morse[5] to follow from a hypothesis of "uniform instability," which is fulfilled on surfaces of negative curvature. Here we shall consider transitive transformations apart from their dynamical origin.

Various examples of transitive transformations are known. Besicovitch[6] recently defined one for the plane. The existence of one for a closed circular region has been mentioned by Kerékjártó[7] as an unsolved problem. The known cases of metric transitivity furnish examples, and likewise the results of Morse and Birkhoff. Birkhoff[2] has stated that the transitive case is probably to be regarded as the general case. The purpose of the present paper is to show that this conjecture can be made precise in the sense of category, provided we confine attention to measure-preserving transformations and thus to establish the general existence of transitive transformations. On the basis of a general lemma concerning metric groups, these results are used to prove a theorem concerning composition of transitive transformations.

Let Ω be any bounded closed region of euclidean r-space ($r \geqq 2$). The topological transformations $T, S, \ldots$ of Ω onto itself constitute a complete metric space $[H]$ with the metric

$$\rho(T,S) = \max_{x \text{ in } \Omega} (|Tx - Sx|, |T^{-1}x - S^{-1}x|).$$

Furthermore, the set of measure-preserving topological transformations of Ω is a closed subset of $[H]$, and is therefore itself a complete metric space which we denote by $[M]$.

The transitive measure-preserving transformations constitute all but a set of first category in $[M]$.

We shall indicate the principal steps in the proof. It suffices to show that the set of transformations such that no image of a neighborhood σ overlaps another neighborhood σ' is nowhere dense in $[M]$, σ and σ' being contained in the interior of Ω. For, by condition (c), every intransitive transformation corresponds to some such pair of neighborhoods; and by taking them as rational spheres, the set of intransitive transformations is represented as a countable sum of nowhere dense sets. The set under consideration being closed, it is sufficient to show that arbitrarily near any given T in $[M]$ one can find a transformation under which σ eventually overlaps σ'. This is accomplished by the following construction. Choose any point x_0 in σ and join any image $T^{k_0}x_0$ to σ' by a thin tube contained in the interior of Ω. Choose x_1 at distance nearly ϵ along this tube, and let it be so chosen that an image $T^{k_1}x_1$ lies very near x_1. This can always be done, since points having this recurrence property are dense.[8] Choose a similar point

x_2 nearly ϵ further along toward σ' and continue until a point x_p in σ' is reached. Consider the finite sequence $x_0, Tx_0, \ldots, T^{k_0}x_0, x_1, Tx_1, \ldots, T^{k_{p-1}}x_{p-1}, x_p$. Join each $T^{k_i}x_i$ to x_{i+1} by a connected neighborhood of diameter less than ϵ, the neighborhoods being taken disjoint and each containing no other points of the sequence. The manner in which the points were chosen makes it clear that this can be done, unless some $T^{k_i}x_i$ coincides with x_i. This case needs to be considered separately, but offers no difficulty. Now within each neighborhood define a transformation S so as to take $T^{k_i}x_i$ into x_{i+1} while leaving everything outside the neighborhoods fixed. For instance, let S be generated by a measure-preserving flow around closed tubes lying in each neighborhood. This S moves no point by more than ϵ, and is such that repeated application of ST takes x_0 into x_p. By choosing ϵ small, the transformation ST can be made to lie arbitrarily close to T, which completes the proof.

It follows at once that transitive measure-preserving transformations of Ω exist, and the proof indicates how they can be defined step by step.

It is an easy matter now to establish the existence of transitive transformations of any region. For we need only map its interior topologically on a bounded open region and consider the space of measure-preserving transformations of the closure of this bounded region which leave all boundary points fixed. The above argument applies to this space without change, and a transitive transformation corresponds to a transitive transformation of the original region.

There exist transitive transformations of any region of euclidean r-space ($r \geqq 2$).

Although the transitive transformations form a residual set in $[M]$, that is, all but a set of first category, they form only a nowhere dense set in $[H]$. For by an arbitrarily small change in a transitive transformation one can make an iterated image of a neighborhood σ lie within a closed subset of σ. Such a transformation is the center of a sphere of intransitive transformations in $[H]$. Thus, in the sense of category, transitivity is to be regarded as the general case among measure-preserving transformations, but is to be considered exceptional among all homeomorphisms.

The spaces $[M]$ and $[H]$ are not only complete metric spaces, but also groups. Furthermore, the group operations of product and inverse are continuous in the metric. Thus $[M]$ and $[H]$ are metric groups.[9] For such systems we have the following result.

If G is any metric group, and R is any residual set in G, then the group product $R \circ R = G$.

Let x be any element of G. The set $x \circ R^{-1}$ is homeomorphic to R, and therefore also residual. Hence the intersection $R \frown x \circ R^{-1}$ is likewise residual and therefore dense. Let r_1 be any one of its elements. Then $r_1 = x \circ r_2^{-1}$, that is, $x = r_1 \circ r_2$, where r_1 and r_2 belong to R. Applying this lemma to the metric group $[M]$, we obtain the following theorem: *Any measure-preserving transformation of* Ω *can be expressed as the product of two transitive measure-preserving transformations. Either factor may be chosen arbitrarily from a set dense in* $[M]$.

In conclusion, it may be mentioned that in the case of three or more dimensions a modified method is available which enables one to prove the existence of transformations transitive in a stronger sense. Namely, every neighborhood has images which are ϵ-dense in Ω, ϵ being arbitrary. The consideration of this kind of transitivity was suggested to me by Dr. S. Ulam. It is hoped to discuss these matters in greater detail subsequently.

* Society of Fellows, Harvard University.

[1] In the weak sense. That is, E can be obtained from an open set by adding and subtracting two first category sets. See C. Kuratowski, *Topologie*, I, Warsaw, 49 (1933).

[2] G. D. Birkhoff, *Acta Math.*, **43**, 1–119 (1922), esp. pp. 111–119.

[3] C. Kuratowski, loc. cit. See also W. Sierpinski, *Fundamenta Mathematicae*, **22**, 276–280 (1934).

[4] G. D. Birkhoff, *Trans. Amer. Math. Soc.*, **18**, 199–300 (1917), esp. p. 286.

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[7] B. de Kerékjártó, *l'Enseignement Math.*, **35**, 297–316 (1936).

[8] Indeed they are all but a set of measure zero. G. D. Birkhoff and P. A. Smith, *Jour. Math.* (Liouville), (9), **7**, 345–379 (1928), esp. p. 355.

[9] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, Chap. 1 (1932).

GROUPS IN WHICH EVERY SET OF INDEPENDENT GENERATORS IS A MAXIMUM SET

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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A group G is said to have a maximum number set of independent generators whenever it has a set of independent generators which involves as many operators as there are prime factors in its order. It has recently been proved[1] that when G involves at least one such set of independent generators its Sylow subgroup whose order is a power of the largest prime number p which divides the order of G is invariant, abelian and of type 1^k. This Sylow subgroup may involve subgroups of order p which are non-invariant under G but when every possible set of independent generators of G involves a maximum number of operators, then every one of these subgroups of order p is invariant under G for if one of these subgroups would not be invariant under G then G would involve a set of independent generators involving an operator of order p which would generate this non-invariant